International Journal of Theoretical Physics, Vol. 33, No. 7, 1994

Decompositions of Measures on Orthoalgebras and Difference Posets

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Received March 16, 1993

We present a general decomposition theorem for elements of an ordered group with respect to a cone. This result enables us to obtain decompositions of finitely additive measures defined on quantum logics, orthoalgebras, or, more generally, on difference posets with values in Dedekind complete lattice ordered groups, with respect to a given cone of measures. In particular, we gain Yosida–Hewitt-type and Lebesgue-type decompositions.

1. INTRODUCTION

In the last decade, decompositions of measures, like those of Yosida and Hewitt (1952) type or Lebesgue type, have received the attention of many authors (Aarnes, 1970; D'Andrea *et al.*, 1991; De Lucia and Dvurečenskij, 1993*a,b*; De Lucia and Morales, 1992; Dvurečenskij, 1991, n.d.; Rüttimann, 1990; Pap, n.d.) interested in finitely additive measures on noncommutative structures. These structures interest specialists working in mathematical foundations of the propositional system of quantum mechanics. Such structures include, for example, quantum logics (= orthomodular posets) presented by Birkhoff and von Neumann, orthoalgebras, originally introduced by Randall and Foulis (1979, 1981), and, more generally, D-posets (= difference posets), recently presented by Kôpka and Chovanec (1994).

In the present paper, we give decompositions of finitely additive measures on quantum logics, orthoalgebras, or D-posets, with values in a Dedekind complete lattice ordered group. The paper is organized so that

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0020-7748/94/0700-1387$07.00/0

first we give a general decomposition theorem for elements of an ordered group with respect to a given cone, and after introducing orthomodular posets, orthoalgebras, and D-posets, we apply the general decomposition theorem to finitely additive measures on the above-mentioned structures.

The method generalizes many earlier results obtained in De Lucia and Dvurečenskij (1993*a*,*b*) and De Lucia and Morales (1992), and it can be applied to different kinds of measures appearing in noncommutative measure theory as well as in fuzzy structures.

2. DECOMPOSITION THEOREM

In the present section, we give a general decomposition theorem which will be applied to decompositions of finitely additive measures on quantum logics, orthoalgebras, D-posets, etc.

Let $(\mathscr{G}, 0, +, \leq_{\mathscr{G}})$ be an *ordered group*, that is, $\mathscr{G}$ is a commutative group written additively, with the neutral element 0, and partial ordering $\leq_{\mathscr{G}}$ such that if $\mu \leq_{\mathscr{G}} \nu$, then $\mu + \xi \leq_{\mathscr{G}} \nu + \xi$ for any $\xi \in \mathscr{G}$. Let $\mathscr{G}_+$ denote the set of all positive elements of $\mathscr{G}$, that is, $\mathscr{G}_+ = \{\mu \in \mathscr{G}: 0 \leq_{\mathscr{G}} \mu\}$. A *cone* of $\mathscr{G}$ is a subset $\mathscr{C}$ of $\mathscr{G}_+$ such that (i) if $\mu_1, \mu_2 \in \mathscr{C}$, then $\mu_1 + \mu_2 \in \mathscr{C}$; and (ii) $0 \in \mathscr{C}$. The cone $\mathscr{C}$ is $\vee$*-closed* if, for any bounded chain C in $\mathscr{C}$, the join $\bigvee^{\mathscr{G}} C := \bigvee^{\mathscr{G}} \{\mu: \mu \in \mathscr{C}\}$ exists in $\mathscr{G}$ and is an element of $\mathscr{C}$.

Let $\mathscr{A}$ be a fixed subset of $\mathscr{G}_+$ containing a cone $\mathscr{C}$. We say that an element $\mu \in \mathscr{A}$ is *singular* with respect to the cone $\mathscr{C}$ if $\nu \leq_{\mathscr{G}} \mu$ for some $\nu \in \mathscr{C}$ implies $\nu = 0$. We denote by $\mathscr{C}^{\#}$ (depending on $\mathscr{A}$, in general) the set of all elements of $\mathscr{A}$ which are singular with respect to $\mathscr{C}$.

Decomposition Theorem. Let $(\mathscr{G}, 0, +, \leq_{\mathscr{G}})$ be a commutative ordered group. Let $\mathscr{A}$ be a subset of $\mathscr{G}_+$ such that if $\xi_1, \xi_2 \in \mathscr{A}$, $\xi_1 \leq_{\mathscr{G}} \xi_2$, then $\xi_2 - \xi_1 \in \mathscr{A}$, and containing a $\vee$-closed cone $\mathscr{C}$. Then for any $\mu \in \mathscr{A}$ there exist two elements $\xi \in \mathscr{C}$ and $\eta \in \mathscr{C}^{\#}$ such that

$$\mu = \xi + \eta \tag{2.1}$$

Proof. Define $\Gamma_\mu = \{\gamma \in \mathscr{C}: \gamma \leq_{\mathscr{G}} \mu\}$. Since the zero element belongs to $\mathscr{C}$, Γ_μ is nonempty. Let $\Gamma_o = \{\gamma_i\}$ be a chain of Γ_μ with respect to $\leq_{\mathscr{G}}$, and define $\gamma_o = \bigvee_i^{\mathscr{G}} \gamma_i$. Due to our assumptions, γ_o is an element of $\mathscr{C}$, and $\gamma_o \leq_{\mathscr{G}} \mu$, so that γ_o is a majorant of Γ_o in Γ_μ. It follows from Zorn's lemma that Γ_μ contains a maximal element ξ which belongs to $\mathscr{C}$ and $\xi \leq_{\mathscr{G}} \mu$.

Put $\eta = \mu - \xi$; clearly $\eta \in \mathscr{A}$. To finish the proof, we show that $\eta \in \mathscr{C}^{\#}$. Let $\gamma \in \mathscr{C}$ be such that $\gamma \leq_{\mathscr{G}} \eta = \mu - \xi$, so that $\gamma + \xi \leq_{\mathscr{G}} \mu$. Because $\gamma + \xi \in \mathscr{C}$, the maximality of ξ in Γ_μ implies $\gamma = 0$. ■

3. ORTHOMODULAR POSETS

An *orthomodular poset* (OMP) is a partially ordered set L with an ordering $\leq$, the least and greatest elements 0 and 1, respectively, an orthocomplementation $\perp: L \to L$ such that

(OMi) $a^{\perp\perp} = a$ for any $a \in L$.
(OMii) $a \vee a^{\perp} = 1$ for any $a \in L$.
(OMiii) If $a \leq b$, then $b^{\perp} \leq a^{\perp}$.
(OMiv) If $a \leq b^{\perp}$ (and we write $a \perp b$), then $a \vee b \in L$.
(OMv) If $a \leq b$, then $b = a \vee (a \vee b^{\perp})^{\perp}$ (orthomodular law).

If in an orthomodular poset L the join of any sequence (any system) of mutually orthogonal elements exist, we say that L is a *σ-orthomodular poset* (a *complete orthomodular poset*). An *orthomodular lattice* is an orthomodular poset L such that, for any $a, b \in L$, $a \vee b$ exists in L (using the de Morgan laws, we see that $a \wedge b$ exists in L, too). A distributive orthomodular lattice is called a *Boolean algebra*. We recall that a lattice L is a Boolean algebra iff for any pair $a, b \in L$ there are three mutually orthogonal elements $a_1, b_1, c \in L$ such that $a = a_1 \vee c$, $b = b_1 \vee c$. For more details concerning orthomodular posets and lattices see, e.g., Kalmbach (1983) and Pták and Pulmannová (1991).

One of the most important cases of orthomodular lattices is the system of all closed subspaces $L(H)$ of a real or complex Hilbert space H, with an inner product $(\cdot, \cdot)$. Here the partial ordering $\leq$ is induced by the natural set-theoretic inclusion, and $M^{\perp} = \{x \in H: (x, y) = 0 \text{ for any } y \in M\}$. Then $L(H)$ is a complete orthomodular lattice, which is not a Boolean algebra, if $\dim H \neq 1$. This structure plays a crucial role in axiomatic foundations of quantum mechanics.

If S is an inner product space (not necessarily complete), denote by $E(S)$ the set of all *splitting subspaces* of S, i.e., the set of all $M \subseteq S$ such that $M + M^{\perp} = S$. Then $E(S)$ is an orthomodular poset which is not necessarily a σ-orthomodular poset. We recall that according to Dvurečenskij (1988) S is complete if and only if $E(S)$ is a σ-orthomodular poset.

4. ORTHOALGEBRAS

An *orthoalgebra* is a set L with two particular elements 0, 1, and with partial binary operation $\oplus: L \times L \to L$ such that for all $a, b, c \in L$ we have:

(OAi) If $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (commutativity).

(OAii) If $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity).

(OAiii) For any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation).

(OAiv) If $a \oplus a$ is defined, then $a = 0$ (consistency).

If the assumptions of (ii) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L.

Let a and b be two elements of an orthoalgebra L. We say that (i) a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined in L; (ii) a is *less than or equal to* b and write $a \leq b$ iff there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ (in this case we also write $b \geq a$); (iii) b is the *orthocomplement* of a iff b is a (unique) element of L such that $b \perp a$ and $a \oplus b = 1$ and it is written as $a^{\perp}$.

If $a \leq b$, for the element c in (ii) with $a \oplus c = b$ we write $c = b \ominus a$, and c is called the *difference* of a in b. It is evident that

$$b \ominus a = (a \oplus b^{\perp})^{\perp} \tag{4.1}$$

Foulis *et al.* (1992) give proofs of the following statements:

Proposition 4.1. Let a, b, and c be elements of an orthoalgebra L. Then

(i) $a \perp b \Leftrightarrow b \perp a$.
(ii) $a \perp a \Rightarrow a = 0$.
(iii) $a \perp 1 \Leftrightarrow a = 0$.
(iv) $a^{\perp\perp} = a$.
(v) $1^{\perp} = 0$ and $0^{\perp} = 1$.
(vi) $a \perp b \Rightarrow a \perp (a \oplus b)^{\perp}, a \oplus (a \oplus b)^{\perp} = b^{\perp}$.
(vii) $a \perp b \Leftrightarrow a \leq b^{\perp}$.
(viii) $a \leq b \Rightarrow b = a \oplus (a \oplus b^{\perp})^{\perp}$.
(ix) $a \oplus b = a \oplus c \Rightarrow b = c$.
(x) $a \oplus b \leq a \oplus c \Rightarrow b \leq c$.
(xi) $0 \leq a \leq 1$, and $\leq$ is a partial ordering on L.
(xii) $a \leq b \Rightarrow b^{\perp} \leq a^{\perp}$.
(xiii) $a \wedge a^{\perp} = 0, a \vee a^{\perp} = 1$.
(xiv) $a \perp b, a \vee b \in L \Rightarrow a \oplus b = a \vee b$.
(xv) $a^{\perp} = 1 \ominus a$.
(xvi) $a \leq b \Leftrightarrow b = a \oplus (b \ominus a)$.
(xvii) $a = a \ominus 0$.
(xviii) $a \leq b \leq c \Leftrightarrow (c \ominus b) \oplus (b \ominus a) = c \ominus a$.
(xix) $a \leq b \leq c \Leftrightarrow (c \ominus a) \ominus (c \ominus b) = b \ominus a$.

We see that if L is an orthomodular poset and $a \oplus b := a \vee b$ whenever $a \perp b$ in L, then L with 0, 1, $\oplus$ is an orthoalgebra. The converse statement does not hold in general, as follows from an example due to R. Wright (Foulis *et al.*, 1992):

Example 4.2. Let $L = \{0, 1, a, b, c, e, f, a^{\perp}, b^{\perp}, c^{\perp}, d^{\perp}, e^{\perp}, f^{\perp}\}$ with $a \oplus b = d \oplus e = c^{\perp}$, $b \oplus c = e \oplus f = a^{\perp}$, $c \oplus d = f \oplus a = e^{\perp}$, $c \oplus e = d^{\perp}$, $a \oplus c = b^{\perp}$, $e \oplus a = f^{\perp}$ is an orthoalgebra that is not an orthomodular poset.

We recall that an orthoalgebra L is an OMP iff $a \perp b$ implies $a \vee b \in L$.

5. DIFFERENCE POSETS

Recently Kôpka and Chovanec (1994) introduced difference posets, (D-posets), which generalize both quantum logics and orthoalgebras, inspired by an investigation of the possibility to introduce fuzzy set ideas into quantum structures models (Kôpka, 1992). In this model, the difference operation is a primary notion from which we derive other usual notions, such as the join of mutually excluded events, which are important for probability calculus.

A *D-poset*, or a *difference poset*, is a partially ordered set L with a partial ordering $\leq$, maximal element 1, and with partial binary operation $\ominus: L \times L \to L$, called a *difference*, such that, for $a, b \in L$, $a \ominus b$ is defined if and only if $a \leq b$, for which the following axioms hold for $a, b, c \in L$:

(DPi) $b \ominus a \leq b$.
(DPii) $b \ominus (b \ominus a) = b$.
(DPiii) $a \leq b \leq c \Rightarrow c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

The following statements have been proved in Kôpka and Chovanec (n.d.):

Proposition 5.1. Let a, b, c, d be elements of a D-poset L. Then:

(i) $1 \ominus 1$ is the minimal element of L; denote it by 0.
(ii) $a \ominus 0 = a$.
(iii) $a \ominus a = 0$.
(iv) $a \leq b \Rightarrow b \ominus a = 0 \Leftrightarrow b = 0$.
(v) $a \leq b \Rightarrow b \ominus a = b \Leftrightarrow a = 0$.
(vi) $a \leq b \leq c \Rightarrow b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.
(vii) $b \leq c, a \leq c \ominus b \Rightarrow b \leq c \ominus a$, and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.
(viii) $a \leq b \leq c \Rightarrow a \leq c \ominus (b \ominus a)$ and $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.

Remark 5.2 (Navara and Pták, n.d.). A poset L with minimal and maximal elements 0 and 1, respectively, and with a partial binary operation

$\ominus: L \times L \to L$ such that for $a, b, c \in L$ we have

(i) $a \ominus 0 = a$
(ii) if $a \le b \le c$, then $c \ominus b \le c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$

is a D-poset.

For any element $a \in L$ we put

$$a^{\perp} := 1 \ominus a$$

Then (i) $a^{\perp\perp} = a$, (ii) $a \le b$; then $b^{\perp} \le a^{\perp}$. Two elements a and b of L are *orthogonal*, and we write $a \perp b$, iff $a \le b^{\perp}$ (iff $b \le a^{\perp}$).

Now we introduce a binary operation $\oplus: L \times L \to L$ such that an element $c = a \oplus b$ in L is defined iff $a \perp b$, and for c we have $b \le c$ and $a = c \ominus b$. The partial operation $\oplus$ is defined correctly because if there exists $c_1 \in L$ with $b \le c_1$ and $a = c_1 \ominus b$, then, by (DPi) and (DPii), we have

$$(1 \ominus (c \ominus b)) \ominus b = 1 \ominus c = (1 \ominus (c_1 \ominus b)) \ominus b = 1 \ominus c_1$$

which implies $c = c_1$. Moreover,

$$a \oplus b = (a^{\perp} \ominus b)^{\perp} = (b^{\perp} \ominus a)^{\perp} \tag{5.1}$$

Indeed, denote by $x = (a^{\perp} \ominus b)^{\perp}$. From (vii) of Proposition 5.1, we conclude that $x = (b^{\perp} \ominus a)^{\perp}$. Therefore, $x^{\perp} = a^{\perp} \ominus b$, which means $a \le x$, analogously, $b \le x$. Calculate

$$x \ominus a = (1 \ominus (b^{\perp} \ominus a)) \ominus a = 1 \ominus b^{\perp} = b$$

where we have used (viii) of Proposition 5.1.

The operation $\oplus$ is commutative (this is evident) and associative: suppose that $y = a \oplus b$ and $z = (a \oplus b) \oplus c$ exist in L. By (DPii) we have

$$(z \ominus a) \ominus (z \ominus y) = y \ominus a$$

$$(z \ominus a) \ominus c = b$$

$$z \ominus a = b \oplus c \in L$$

$$z = a \oplus (b \oplus c) \in L$$

so that (OAi)–(OAiii) are valid.

It is evident that any orthoalgebra L is a D-poset when a difference $\ominus$ is defined by (4.1). Indeed, (DPi) and (DPii) are trivially satisfied, and (DPiii) follows from (xix) of Proposition 4.1.

By Navara and Pták (n.d.) we conclude that a D-poset L with 0, 1, and $\oplus$ defined by (5.1) is an orthoalgebra if and only if $a \le 1 \ominus a$ implies

$a = 0$. Therefore, it is not hard to give many examples of D-posets which are not orthoalgebras.[2]

An interesting model, MV-*algebras* introduced by Mundici (1986), gives an example of D-posets.

Example 5.2. An MV-algebra is an algebra $(M, \oplus, \odot, \star, 0, 1)$, where $0, 1 \in M$, $\oplus$ and $\odot$ are binary operations, and $\star$ is a unary operation, such that, for all $x, y, z \in M$:

(i) $x \oplus y = y \oplus x$.
(ii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.
(iii) $x \oplus 0 = x$.
(iv) $x \oplus 1 = 1$.
(v) $(x^\star)^\star = x$.
(vi) $0^\star = 1$.
(vii) $x \oplus x^\star = 1$.
(viii) $(x^\star \oplus y)^\star \oplus y = (x \oplus y^\star)^\star \oplus x$.
(ix) $x \odot y = (x^\star \oplus y^\star)^\star$.

An MV-algebra M is a distributive lattice when $x \vee y := (x \odot y^\star) \oplus y$ and $x \wedge y := (x \oplus y^\star) \odot y$ with 0 and 1 as minimal and maximal elements (the partial ordering $\leq$ is defined in M via $x \leq y$ iff $x \vee y = y$). If for $x \leq y$ we define $y \ominus x := (x \oplus y^\star)^\star$, then M with $1, \ominus, \leq$ is a D-poset.

Example 5.3 (Kôpka, 1992). Let $F = [0, 1]^\Omega$, where $\Omega \neq \emptyset$. Elements of F are called fuzzy sets of Ω. For two fuzzy sets f and g of F we write $f \leq g$ iff $f(\omega) \leq g(\omega)$ for every $\omega \in \Omega$. Let Φ: $[0, 1] \to [0, \infty)$ be an injective increasing continuous function such that $\Phi(0) = 0$. If we define for $f \leq g$

$$(g \ominus f)(\omega) = \Phi^{-1}(\Phi(g(\omega)) - \Phi(f(\omega))), \qquad \omega \in \Omega$$

then F with $1, \ominus, \leq$ is a D-poset.

As a case of Φ we choose $\Phi(t) = t$, or $\Phi(t) = t^2$, for $t \in [0, 1]$.

6. LATTICE ORDERED GROUPS

Let V with $+, 0, \vee, \wedge, \leq$ be a nontrivial *lattice ordered group*, such that if $(V, +, 0)$ is a nontrivial commutative additive group with the neutral element 0, $(V, \vee, \wedge, \leq)$ is a lattice with respect to a partial ordering $\leq$,

[2]Such an example is the set of all effects of a Hilbert space H, $\mathscr{E}(H)$, i.e., of all Hermitian operators A on H with $O \leq A \leq I$, where O and I are the nonzero and identity operators, respectively, on H, which are important for unsharp measurements of quantum mechanics. We recall that $\mathscr{E}(H)$ is a D-poset (Kôpka and Chovanec, 1994) but evidently not an orthoalgebra.

and group and lattice operations are related as follows:

$$x \leq y \;\Rightarrow\; x + z \leq y + z, \qquad \forall z \in V$$

Then V is a distributive lattice for which minimal and maximal elements do not exist in V. For example, any Riesz space is a lattice ordered group. We recall that one of the most important examples of lattice ordered groups for measure theory is the additive group of all real numbers with the natural ordering of real numbers.

A lattice ordered group is said to be *Dedekind complete* if, for any nonvoid majorized subset B of V, $\bigvee B := \bigvee \{b : b \in B\}$ exists in V. If the former holds for any countable B, V is said to be *σ-Dedekind complete.*

A space V is *Archimedean* if, for some $x, y \in V$ with $nx \leq y$ for every integer n, we have $x \leq 0$; we conclude that $\mu(0) = 0$. It is well known that if V is Dedekind complete, it is σ-complete, and any σ-complete space is Archimedean. For more information on lattice ordered groups see, e.g., Birkhoff (1967).

A nonempty set D of V is *directed downward* (*upward*), and we write $D \downarrow (D \uparrow)$ if for any $x, y \in D$ there exists $z \in D$ such that $z \leq x, z \leq y$ $(z \geq x, z \geq y)$. Two downward-directed sets $\{x_t : t \in T\}$ and $\{y_t : t \in T\}$ indexed by the same index set T are called *equidirected* if, for any $s, t \in T$, there exists $v \in T$ such that $x_v \leq x_s$ and $x_v \leq x_t$ as well as $y_v \leq y_s$ and $y_v \leq y_t$. A similar definition holds for upward-directed sets.

Let $x \in V$ and $D \subset V$. We say that $D \uparrow x$ if $D \uparrow$ and $x = \bigvee D$. Dually we define $D \downarrow x$, i.e., $D \downarrow$ and $x = \bigwedge D$. If $\{f_t\}$ and $\{g_t\}$ are equidirected, then

$$\{f_t\} \uparrow f, \qquad \{g_t\} \uparrow g \;\Rightarrow\; \{f_t + g_t\} \uparrow f + g \tag{6.1}$$

$$\{f_t\} \downarrow f, \qquad \{g_t\} \downarrow g \;\Rightarrow\; \{f_t + g_t\} \downarrow f + g \tag{6.2}$$

Finally, for any $x \in V$: $x^{\perp} = x \vee 0$, $x^- = (-x) \vee 0$, $|x| = x^+ + x^-$. We have, for all $x, y \in L$, (i) $x = x^+ - x^-$, (ii) $|x| = 0$ iff $x = 0$, (iii) $|x + y| \leq |x| + |y|$. By V_+ we denote the set of all positive elements of V, i.e., $V_+ = \{x \in V : x \geq 0\}$.

7. MEASURES ON DIFFERENCE POSETS

Throughout the rest of this paper, by L and V we mean a D-poset with $1, \leq, \ominus$, for which the partial binary operation $\oplus : L \times L \to L$ is defined by (5.1), and a lattice ordered group with $+, 0, \vee, \wedge, \leq$, respectively. Define the following natural ordering $\leq_n$ on V^L: $\mu_1 \leq_n \mu_2$ iff $\mu_1(a) \leq \mu_2(a)$ for any $a \leq L$.

We say that an element $\mu \in V^L$ is a *finitely additive measure* if $\mu(a \oplus b) = \mu(a) + \mu(b)$ whenever $a \oplus b$ is defined in L. Then $\mu(0) = 0$, and $\mu(a^\perp) = \mu(1) - \mu(a)$, $a \in L$. If $\mu: L \to V_+$, then $a \le b$ implies $\mu(a) \le \mu(b)$.

We recall that if μ is a finitely additive measure on L, then $\mu(b \ominus a) = \mu(b) - \mu(a)$ whenever $a \le b$. Conversely, an element $\mu \in V^L$ is a finitely additive measure iff $\mu(b \ominus a) = \mu(b) - \mu(a)$ whenever $a \le b$; this follows easily from (5.1).

To define σ-additive and completely additive measures on L, we introduce the following notions.

Let $F = \{a_1, \ldots, a_n\} \subseteq L$. Recursively we define for $n \ge 3$

$$a_1 \oplus \cdots \oplus a_n := (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n \tag{7.1}$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in L. From the associativity of $\oplus$ in D-posets we conclude that (7.1) is correctly defined. Definitionally we put $a_1 \oplus \cdots \oplus a_n = a_1$ if $n = 1$ and $a_1 \oplus \cdots \oplus a_n = 0$ if $n = 0$. Then for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ and any k with $1 \le k \le n$ we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n} \tag{7.2}$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n) \tag{7.3}$$

We say that a finite set $F = \{a_1, \ldots, a_n\}$ of L is $\bigoplus$*-orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists in L. In this case we say that F has a $\bigoplus$*-sum*, $\bigoplus_{i=1}^n a_i$, defined via

$$\bigoplus_{i=1}^n a_i = a_1 \oplus \cdots \oplus a_n \tag{7.4}$$

It is clear that two elements a and b of L are orthogonal, i.e., $a \perp b$, iff $\{a, b\}$ is $\bigoplus$-orthogonal.

An arbitrary subset G of L is $\bigoplus$*-orthogonal* if every finite subset F of G is $\bigoplus$-orthogonal. If G is $\bigoplus$-orthogonal, so is any of its subsets. An $\bigoplus$-orthogonal subset $G = \{x_i : i \in I\}$ of L has a $\bigoplus$*-sum* in L, written as $\bigoplus_{i \in I} a_i$, if in L there exists the join

$$\bigoplus_{i \in I} a_i := \bigvee_F \bigoplus_{i \in F} a_i \tag{7.5}$$

where F runs over all finite subsets in I.

It is evident that if $G = \{a_1, \ldots, a_n\}$ is $\bigoplus$-orthogonal, then the $\bigoplus$-sums defined by (7.4) and (7.5) coincide.

We say that a D-poset L is a *complete D-poset* (*σ-D-poset*) if, for any $\bigoplus$-orthogonal subset (any countable $\bigoplus$-orthogonal subset) G of L, there exists the $\bigoplus$-sum in L. It is straightforward to verify that a D-poset L is a σ-D-poset if, for any sequence $\{a_i\}$ in L with $a_1 \le a_2 \le \cdots$, the join $\bigvee_{i=1}^\infty a_i$ exists in L.

Introducing an order convergence, we are able to define a σ-additive measure $\mu: L \to V$.

We say that a sequence $\{x_n\}$ in V is *order convergent* to an element $x \in V$, and we write $x_n \to x$, if there exist a downward-directed sequence $\{p_n\} \downarrow 0$ such that $|x_n - x| \le p_n$ for every n. If $x_n \to x$ and $x_n \to y$, then $x = y$, moreover, if $x_n \uparrow x$ ($x_n \downarrow x$), then $x_n \to x$.

An element $\mu \in V^L$ is a *σ-additive measure* if, for any $\oplus$-orthogonal sequence $\{a_n\}$ with the $\oplus$-sum $a = \bigoplus_{n=1}^{\infty} a_n$ in L, we have $\sum_{i=1}^{n} \mu(a_i) \to \mu(a)$. We recall that an element $\mu \in V_+^L$ is a σ-additive measure iff $\mu(b \ominus a) = \mu(b) - \mu(a)$ whenever $a \le b$, and for any sequence $\{a_n\}$ with $a_1 \le a_2 \le \cdots$, and $a = \bigvee_{n=1}^{\infty} a_n$ existing in L, we have $\mu(a) = \bigvee_{n=1}^{\infty} \mu(a_n)$.

A mapping $\mu \in V_+^L$ is said to be a *positive completely additive measure* on L if, for any $\oplus$-orthogonal system $\{a_i: i \in I\}$, for which the $\oplus$-sum $\bigoplus_{i \in I} a_i$ exists in L, we have for any finite subset F of I,

$$\left|\mu\left(\bigoplus_{i \in I} a_i\right) - \sum_{i \in F} \mu(a_i)\right| \le b_F \tag{7.6}$$

where $\{b_F\} \downarrow 0$ and $b_{F_1} \le b_{F_2}$ whenever $F_2 \subseteq F_1$. Due to (6.2), (7.6) is defined correctly, and we shall write $\mu(\bigoplus_{i \in I} a_i) = \sum_{i \in I} \mu(a_i)$.

If the index set I in (7.6) is only countable, we say that μ is a *positive σ-additive measure*, and we write $\mu(\bigoplus_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} \mu(a_i)$. It is clear that this notion coincides with the above-defined one for positive measures.

Since any Dedekind complete lattice ordered group is Archimedean, we conclude that $\mu(0) = 0$. Indeed, for any finite subset F of I with $|\mu(\bigoplus_{i \in I} a_i) - \sum_{i \in F} \mu(a_i)| \le b_F$, where $a_i = 0$ for any $i \in I$, we have $(\text{card } F - 1)\ |\mu(0)| \le b_F \downarrow 0$, so that $\mu(0) = 0$.

Moreover, any completely additive measure is σ-additive, and any σ-additive measure is finitely additive.

We denote by $a(L, V)_+$, $\sigma a(L, V)_+$, and $ca(L, V)_+$ the sets of all positive finitely additive, σ-additive, and completely additive measures, respectively, from V_+^L.

It is not hard to prove that a positive additive measure μ on L is σ-additive, or completely additive, iff

$$\left\{\sum_{i=1}^{n} \mu(a_i)\right\} \uparrow \mu\left(\bigoplus_{i=1}^{\infty} a_i\right) \tag{7.7}$$

or

$$\left\{\sum_{i \in F} \mu(a_i)\right\}_F \uparrow \mu\left(\bigoplus_{i \in I} a_i\right) \tag{7.8}$$

where F runs over all finite subsets of I whenever $\bigoplus_{i=1}^{\infty} a_i$ or $\bigoplus_{i \in I} a_i$, respectively, exists in L.

8. DECOMPOSITIONS OF MEASURES

In the present section, we give the main results of the paper. If we define $\mathscr{G}$ as the set of all finitely additive measures from a D-poset L into a lattice ordered group V, $\mathscr{G}$ becomes an ordered group when $\leq_{\mathscr{G}} := \leq_n$ and $(\mu_1 + \mu_2)(a) := \mu_1(a) + \mu_2(a)$, $a \in L$. Using the Decomposition Theorem, we are ready to prove many decomposition theorems for positive finitely additive measures on D-posets, in particular, Yosida–Hewitt-type and Lebesgue-type decompositions.

Theorem 8.1. Let L be a D-poset and V a Dedekind complete lattice group. Let $\mathscr{C}$ be a $\vee$-closed cone of $a(L, V)_+$. Then for any $\mu \in a(L, V)_+$ there exist two elements $\xi \in \mathscr{C}^{\#}$ such that

$$\mu = \xi + \eta \tag{8.1}$$

Proof. This follows from the Decomposition Theorem upon setting $\mathscr{A} = \mathscr{G}$. ■

Theorem 8.2. Every positive finitely additive measure μ from a D-poset L into a Dedekind complete lattice ordered group V can be expressed as a sum $\mu = \xi + \eta$, where ξ is a positive completely additive measure from V^L, and η is a finitely additive measure such that if $\zeta \leq_n \eta$, $\zeta \in ca(L, V)_+$. then $\zeta = 0$.

Proof. Define the set $\mathscr{C} = ca(L, V)_+$. We show that $\mathscr{C}$ is a $\vee$-closed cone.

First, let $\{\gamma_i\}$ be a chain in $a(L, V)_+$ with a bound $\gamma \in a(L, V)_+$ with respect to the natural ordering $\leq_n$, and define

$$\gamma_o(c) = \bigvee_i \gamma_i(c), \qquad c \in L \tag{8.2}$$

Since $0 \leq \gamma_i(c) \leq \gamma_i(1) \leq \gamma(1)$, the Dedekind completeness of V implies that $\gamma_0(c)$ is defined correctly on L. Moreover, γ_o is finitely additive. Indeed, let $a \oplus b$ be defined in L. Then $\{\gamma_i(a)\}$ and $\{\gamma_i(b)\}$ are equidirected, and $\gamma_i(a) \uparrow \gamma_0(a)$, $\gamma_i(b) \uparrow \gamma_0(b)$. By (6.1) we conclude that

$$\gamma_o(a \oplus b) = \gamma_i(a \oplus b) \uparrow = (\gamma_i(a) + \gamma_i(b)) \uparrow = \gamma_i(a) \uparrow + \gamma_i(b) \uparrow = \gamma_o(a) + \gamma_o(b)$$

Therefore,

$$\gamma_0 = \bigvee_i^{\mathscr{G}} \gamma_i \tag{8.3}$$

where $\bigvee^{\mathscr{G}}$ is taken in $\mathscr{G}$, and conversely, if $\gamma'_o = \bigvee_i^{\mathscr{G}} \gamma_i$, then $\gamma'_o = \gamma_o$.

Now we claim to show that $\gamma_o \in ca(L, V)_+$ if $\{\gamma_i\}$ is a bounded chain in $\mathscr{C}$. Let $a = \bigoplus_{i \in I} a_i$ exist in L. Then, for any finite subsets F of I, we have

$$\begin{aligned} 0 &\le \gamma_o(a) - \sum_{i\in F} \gamma_o(a_i) \\ &= \gamma_o\left(a \ominus \left(\bigoplus_{i\in F} a_i\right)\right) \\ &= \left(\gamma_o\left(a \ominus \left(\bigoplus_{i\in F} a_i\right)\right) - \gamma_i\left(a \ominus \left(\bigoplus_{i\in F} a_i\right)\right)\right) + \gamma_i\left(a \ominus \left(\bigoplus_{i\in F} a_i\right)\right) \\ &\le p_i + b_F^i \end{aligned}$$

where $\{p_i\} \downarrow 0$ and $\{b_F^i\}_F \downarrow 0$, and F is a finite subset of I. Then

$$0 \le \gamma_o(a) - \bigvee_F \sum_{i\in F} \gamma_o(a_i) \le p_i \downarrow 0$$

so that $\gamma_o(a) = \sum_{i\in I} \gamma_o(a_i) \in \mathscr{C}$, which means that $\mathscr{C}$ is a $\vee$-closed cone.

Now we can apply Theorem 8.1 to obtain the assertion in question. ■

Theorem 8.3. Every positive finitely additive measure μ on a D-poset L with values in a Dedekind complete lattice ordered group V can be expressed as a sum $\mu = \xi + \eta$, where $\xi \in \sigma a(L, V)_+$ and $\eta \in \sigma a(L, V)_+^{\#}$.

Proof. The proof is identical to the proof of Theorem 8.2, if we use the cone $\mathscr{C} = \sigma a(L, V)_+$. ■

Theorem 8.4. Every positive σ-additive measure μ on a D-poset L with values in a Dedekind complete lattice ordered group V can be expressed as a sum $\mu = \xi + \eta$, where $\xi \in ca(L, V)_+$ and η is a positive σ-additive measure such that if $\zeta \le_n \eta$, $\zeta \in ca(L, V)_+$, then $\zeta = 0$.

Proof. This follows from Theorem 8.1 and the Decomposition Theorem if we put $\mathscr{C} = ca(L, V)_+$ and $\mathscr{A} = \sigma a(L, V)_+$. ■

Remark 8.5. Theorems 8.2 and 8.3 have been proved in De Lucia and Dvurečenskij (1993*a*) and De Lucia and Morales (1992). They are analogs of the classical Yosida and Hewitt (1952) decomposition as well as Theorem 8.4. In De Lucia and Morales (1992) the component η from decomposition Theorems 8.2 and 8.3 is said to be a *weakly purely additive measure* and a *purely additive measure*, and η in Theorem 8.5 is called a *purely σ-additive measure.*

Let $\mathscr{P}$ be a nonvoid subset of a D-poset L. A positive finitely additive measure $\mu \in a(L, V)_+$ is said to be *$\mathscr{P}$-regular* if

$$\mu(a) = \bigvee \{\mu(b): b \leq a, b \in \mathscr{P}], \qquad a \in L \tag{8.4}$$

We denote by $a_{\mathscr{P}}(L, V)_+$ the set of all $\mathscr{P}$-regular elements from $a(L, V)_+$.

Theorem 8.6. Let $\mathscr{P}$ be a nonempty set of a D-poset L such that if $a, b \in \mathscr{P}$, then $a \vee b$ exists in L and belongs to $\mathscr{P}$. Then every element $\mu \in a(L, V)_+$ can be expressed as a sum $\mu = \xi + \eta$, where ξ is a $\mathscr{P}$-regular positive finitely additive measure and $\eta \in a_{\mathscr{P}}(L, V)_+^{\#}$.

Proof. We claim to show that the set $\mathscr{C} = a_{\mathscr{P}}(L, V)_+$ is a $\vee$-closed cone. It is clear that $0 \in \mathscr{C}$. Let $\mu_1, \mu_2 \in \mathscr{C}$, and let a be a given element of L. Define $b_F := \bigvee \{b: b \in F\}$, where F is any finite subset of the set $\{b \in \mathscr{P}: b \leq a\}$. Then $\{\mu_i(b_F)\}_F \uparrow \mu_i(a)$, $i = 1, 2$, and $\{\mu_1(b_F)\}_F$ and $\{\mu_2(b_F)\}_F$ are equidirected upward. Using (6.1), we have

$$(\mu_1 + \mu_2)(a) = \mu_1(a) + \mu_2(a) = \mu(b_F) \uparrow + \mu_2(b_F) \uparrow = (\mu_1 + \mu_2)(b_F) \uparrow$$

so that $\mu_1 + \mu_2 \in \mathscr{C}$.

The $\vee$-closedness of $\mathscr{C}$ can be proved as follows: let γ_o be defined by (8.2), where $\{\gamma_i\}$ is a bounded chain in $\mathscr{C}$. Then, for any $a \in L$ and $b \in \mathscr{P}$ with $b \leq a$, we have

$$\gamma_o(a) = \bigvee_i \gamma_i(a) = \bigvee_i \bigvee_b \gamma_i(b) = \bigvee_b \bigvee_i \gamma_i(b) = \bigvee_b \gamma_o(b)$$

which entails that $\gamma_o \in \mathscr{C}$.

The rest follows from Theorem 8.1. ■

9. LEBESGUE DECOMPOSITIONS

Below we present two Lebesgue-type decompositions which generalize those in De Lucia and Dvurečenskij (1993*a*) and De Lucia and Morales (1992).

Let W be another lattice ordered group, and let $\mu \in a(L, V)_+$, $\lambda \in a(L, V)_+$ be given. We say that (i) μ is *λ-continuous*, and we write $\mu \ll_\epsilon \lambda$, if for every $\epsilon > 0$, $\epsilon \in V_+$, there is $\delta > 0$, $\delta \in W_+$ such that every $a \in L$ with $\lambda(a) < \delta$ implies $\mu(a) < \epsilon$; (ii) μ is *dominated* by λ, and we write $\mu \ll \lambda$, if $\lambda(a) = 0$ implies $\mu(a) = 0$; (iii) μ is *singular*, and we write $\mu \perp \lambda$, if there exists $a \in L$ such that $\mu(a^\perp) = 0$ and $\lambda(a) = 0$; (iv) μ is *λ-singular* if, whenever $\gamma \in a(L, V)_+$, $\gamma \ll_\epsilon \lambda$, and $\gamma \leq_n \lambda$, then $\gamma = 0$.

Theorem 9.1. Let V have the property that for any $u > 0$, $u \in V$ there exist $u_1, u_2 \in V$, $u_1, u_2 > 0$, with $u_1 + u_2 = u$. Let $\mu \in a(L, V)_+$ and $\lambda \in a(L, W)_+$. Then μ can be expressed in the form $\mu = \xi + \eta$, where $\xi, \eta \in a(L, V)_+$, $\xi \ll_\epsilon \lambda$, and η is λ-singular.

Proof. Let us define $\mathscr{C} = \{\gamma \in a(L, V)_+ : \gamma \ll_\epsilon \lambda\}$. Then $\mathscr{C}$ is a $\vee$-closed cone of $a(L, V)_+$. Indeed, $0 \in \mathscr{C}$, and let $\gamma_1, \gamma_2 \in \mathscr{C}, \epsilon \in V_+$. There are two $\epsilon_1, \epsilon_2 > 0$ with $\epsilon = \epsilon_1 + \epsilon_2$. We can find $\delta_1 > 0$ and $\delta_2 > 0$ in W such that $\lambda(a_1) < \delta_1$ implies $\gamma(a_1) < \epsilon_1$ and $\lambda(a_2) < \delta_2$ implies $\gamma(a_2) < \epsilon_2$. Put $\delta = \delta_1 \wedge \delta_2$ and let $\lambda(a) < \delta$. Then $\gamma_1(a) + \gamma_2(a) < \epsilon_1 + \epsilon_2 = \epsilon$.

Now let $\{\gamma_i\}$ be bounded chain in $\mathscr{C}$, and define γ_o via (8.2). Given γ_{i_o} we find $\delta > 0$ in W such that $\lambda(a) < \delta$ implies $\gamma_{i_o}(a) < \epsilon$. Then

$$\gamma(a) = \gamma(a) - \gamma_{i_o}(a) + \gamma_{i_o}(a) < p_{i_o} + \epsilon$$

where $\{p_i\} \downarrow 0$, because $\{\gamma(a) - \gamma_{i_o}(a)\}_i \downarrow 0$ uniformly in $a \in L$. Therefore, $\gamma(a) < \epsilon$.

To obtain the desired decomposition, we apply Theorem 8.1. ■

Theorem 9.2. For any pair of finitely additive measures $\mu, \lambda \in a(L, V)_+$ there exist two elements ξ and η in $a(L, V)_+$ such that

$$\mu = \xi + \eta, \qquad \xi \ll \lambda \tag{9.1}$$

and $\eta \wedge \lambda = 0$.

Proof. The set $\mathscr{C}_\lambda = \{\xi \in a(L, V)_+ : \xi \ll \lambda\}$ is a $\vee$-closed cone in $a(L, V)_+$. Applying Theorem 8.1, we obtain the decomposition (9.1), where $\eta \in \mathscr{C}_\lambda^\#$.

Suppose now that κ is an element of $a(L, V)_+$ such that $\kappa \leq_n \eta$ and $\kappa \leq_n \lambda$. Then $\kappa \in \mathscr{C}_\lambda$ and from the basic property of the set $\mathscr{C}_\lambda^\#$ we conclude that $\kappa = 0$, i.e., $0 = \eta \wedge \lambda$. ■

10. CONCLUDING REMARKS

It is worth noting that in some particular cases, $\mathscr{C}$ can consist only of the zero function, and in this case, the decomposition (2.1) is trivial, since $\mathscr{C}^\# = a(L, V)_+$. On the other hand, one can find other important cones. For example, such a situation occurs when $L = E(S)$ is the set of all splitting subspaces for any incomplete inner product space S, and $V = \mathbb{R}$, because, by Dvurečenskij (1993*b*), S is complete iff $ca(E(S), \mathbb{R})_+ \neq \{0\}$. However, in this case, the set of all $\mathscr{P}$-regular finitely additive measures, where $\mathscr{P}$ is the set of all finite-dimensional subspaces of S, is very rich (Dvurečenskij, 1991, 1993*b*).

It is well known that the space of all bounded, finitely additive, real-valued measures on a Boolean algebra is a lattice ordered group (in fact, a Riesz space). For quantum logics, orthoalgebras, or D-posets, this statement is not true, in general, as the following assertion says:

Proposition 10.1. Let H be a finite-dimensional Hilbert space of dimension at least three. Then the set of all bounded finitely additive measures on $L(H)$ is not a lattice with respect to the natural ordering. In particular, $\mu \wedge \eta$ exists as a bounded finitely additive measure on $L(H)$ if and only if either $\mu \leq_n \eta$ or $\eta \leq_n \mu$.

Proof. Due to Gleason's theorem (Dvurečenskij, 1993*a*) there exists a one-to-one correspondence between the set of all bounded, real-valued, finitely additive measures $\{\mu\}$ on $L(H)$ and the set of all Hermitian operators $\{T\}$ on H determined via

$$\mu(M) = \operatorname{tr}(TP_M), \qquad M \in E(S) \tag{10.1}$$

where P_M denotes the orthoprojector from H onto M.

Let μ_T, μ_S, and μ_U be measures determined by Hermitian operators T, S, and U on H via (10.1). It is easy to verify that $\mu_T \leq_n \mu_U$ iff $T \leq U$ [$T \leq U$ means $(Tf, f) \leq (Uf, f)$ for any $f \in H$], so that $\mu_T \wedge \mu_S = \mu_U$ exists iff $T \wedge S = U$. According to Kadison's result (Luxemburg and Zaanen, 1971, 58.4), $T \wedge S$ exists as a Hermitian operator on H iff T and S are comparable; consequently, $\mu_T \wedge \mu_S$ exists iff μ_T and μ_S are comparable, i.e., $\mu_T \leq_n \mu_S$ or $\mu_S \leq_n \mu_T$). ■

From the former assertion it follows that it is not possible to use all the methods that work in lattice ordered groups; in particular, we did not prove the uniqueness of decomposition for measures on quantum logic, orthoalgebras, or D-posets. On the other hand, Aarnes (1970) and Dvurečenskij (1991, 1993*b*) proved decomposition theorem together with uniqueness for $L(H)$ and $E(S)$ using the deep Gleason theorem.

We recall that it is also possible to study measures on difference posets with values in lattice ordered semigroups V. However, posing natural conditions to V such as V is cancelative, i.e., $x + y = z + y$, then $x = z$, and V is naturally ordered, i.e., if $x \leq y$, then $x + z = y$ for some $z \in V$, we obtain, by Birkhoff (1967), that V is a positive cone of an ordered group, which means that we obtain all results of the present paper.

To finish this section, we note that the general Decomposition Theorem enables us to derive another decomposition of measures which are defined not only on quantum logics, orthoalgebras, or D-posets. Such possible structures include, for example, measures on fuzzy quantum structures (Dvurečenskij and Riečan, 1991) and generalized measures (Klement and Weber, 1991).

ACKNOWLEDGMENT

This research was supported by grants G-368 and G-367 of the Slovak Academy of Sciences, Slovakia.

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